Homework 2 Solutions

Based on the third edition of *Abstract Algebra* of Dummit and Foote.

1.6 Homomorphisms and Isomorphisms

Section 1.6, Exercise 17

Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

Solution. Let $\iota : G \to G$ be the inversion map, $\iota(g) = g^{-1}$.

Suppose first that ι is a homomorphism. For arbitrary $x, y \in G$, the homomorphism property gives

$$(xy)^{-1} = \iota(xy) = \iota(x)\iota(y) = x^{-1}y^{-1}.$$

On the other hand, we have $(xy)^{-1} = y^{-1}x^{-1}$. Hence $x^{-1}y^{-1} = y^{-1}x^{-1}$ which yields $yx = xy$. Hence G is abelian.

Conversely, suppose G is abelian. Then for all $x, y \in G$,

$$\iota(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \iota(x)\iota(y).$$

Thus ι is a homomorphism. $\square$

Section 1.6, Exercise 18

Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

Solution. Define $q : G \to G$ by $q(g) = g^2$. Suppose q is a homomorphism. Then for every $x, y \in G$,

$$xyxy = (xy)^2 = q(xy) = q(x)q(y) = x^2y^2 = xxyy.$$

Cancel x on the left and then cancel y on the right to obtain $yx = xy$. Thus every two elements of G commute, so G is abelian.

Conversely, if G is abelian, then

$$q(xy) = (xy)^2 = xyxy = xxyy = x^2y^2 = q(x)q(y)$$

for all $x, y \in G$. Hence q is a homomorphism. $\square$

Section 1.6, Exercise 23

Let G be a finite group which possesses an automorphism σ (cf. Exercise 20) such that $\sigma(g) = g$ if and only if $g = 1$. If σ^2 is the identity map from G to G, prove that G is abelian (such an automorphism σ is called fixed point free of order 2). [Show that every element of G can be written in the form $x^{-1}\sigma(x)$ and apply σ to such an expression.]

Solution. Consider the map

$$f : G \longrightarrow G, \qquad f(x) = x^{-1}\sigma(x).$$

We first prove that f is injective. If $f(x) = f(y)$, then $x^{-1}\sigma(x) = y^{-1}\sigma(y)$. This gives

$$yx^{-1} = \sigma(y)\sigma(x)^{-1} = \sigma(yx^{-1}).$$

Thus yx^{-1} is fixed by σ. By the fixed-point-free hypothesis, $yx^{-1} = 1$, so $y = x$. Therefore f is injective. Since G is finite, f is also surjective. Consequently, every $g \in G$ has the form

$$g = x^{-1}\sigma(x)$$

for some $x \in G$. Using $\sigma^2 = \mathrm{id}_G$, we obtain

$$\sigma(g) = \sigma(x)^{-1}\sigma^2(x) = \sigma(x)^{-1}x = \left(x^{-1}\sigma(x)\right)^{-1} = g^{-1}.$$

Hence $\sigma(g) = g^{-1}$ for every $g \in G$. Now take arbitrary $a, b \in G$. Because σ is a homomorphism,

$$a^{-1}b^{-1} = \sigma(a)\sigma(b) = \sigma(ab) = (ab)^{-1} = b^{-1}a^{-1}.$$

Taking inverses gives $ba = ab$. Thus G is abelian. □

2.1 Subgroups

Section 2.1, Exercise 3

Show that the following subsets of the dihedral group D_8 are actually subgroups:

(a) $\{1, r^2, s, sr^2\}$,

(b) $\{1, r^2, sr, sr^3\}$.

Solution. Use the presentation $D_8 = \langle r, s \mid r^4 = s^2 = 1,\ sr = r^{-1}s\rangle$. The relation $sr^k = r^{-k}s$ gives $sr^2 = r^{-2}s = r^2 s$, so r^2 commutes with s (and clearly with r). Thus r^2 is central and has order 2.

(a) The elements r^2 and s are commuting involutions. Therefore

$$\left\langle r^2, s\right\rangle = \{1, r^2, s, r^2 s\} = \{1, r^2, s, sr^2\}.$$

As a subgroup generated by elements of D_8, this set is a subgroup of D_8.

(b) The element sr is also an involution, since $(sr)^2 = srsr = r^{-1}r = 1$. Because r^2 is central, it commutes with sr, and $r^2(sr) = sr^3$. Consequently

$$\left\langle r^2, sr\right\rangle = \{1, r^2, sr, r^2(sr)\} = \{1, r^2, sr, sr^3\},$$

which is therefore a subgroup of D_8.

□

Section 2.1, Exercise 4

Give an explicit example of a group G and an infinite subset H of G that is closed under the group operation but is not a subgroup of G.

Solution. Take the additive group $G = (\mathbb{Z}, +)$ and let

$$H = \{0, 1, 2, 3, \ldots\}.$$

The set H is infinite, and if $a, b \in H$, then $a + b \in H$, so it is closed under the group operation. However, $1 \in H$ while its additive inverse $-1 \notin H$. Hence H is not closed under inverses and is not a subgroup of G. $\square$

Section 2.1, Exercise 6

Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the torsion subgroup of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

Solution. Let

$$T = \{g \in G : |g| < \infty\}.$$

The identity belongs to T. If $x, y \in T$, let $|x| = m$ and $|y| = n$, and put $L = \mathrm{lcm}(m, n)$. Because G is abelian,

$$(xy^{-1})^L = x^L (y^{-1})^L = 1 \cdot 1 = 1.$$

Thus xy^{-1} has finite order, so $xy^{-1} \in T$. The subgroup criterion shows that $T \leq G$.

For a non-abelian counterexample, let $G = \mathrm{GL}_2(\mathbf{R})$. Consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then $|A| = 4$ and $|B| = 3$. But $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which has infinite order. $\square$

Section 2.1, Exercise 8

Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup if and only if either $H \subseteq K$ or $K \subseteq H$.

Solution. If $H \subseteq K$, then $H \cup K = K$, which is a subgroup. Similarly, if $K \subseteq H$, then $H \cup K = H$ is a subgroup.

Conversely, suppose $H \cup K$ is a subgroup. If neither subgroup contains the other, choose

$$h \in H \setminus K, \qquad k \in K \setminus H.$$

Both h and k lie in $H \cup K$, so the subgroup assumption implies $hk \in H \cup K$. If $hk \in H$, then $k = h^{-1}(hk) \in H$, a contradiction. If $hk \in K$, then $h = (hk)k^{-1} \in K$, also a contradiction. Therefore one of the containments $H \subseteq K$ or $K \subseteq H$ must hold. $\square$

Section 2.1, Exercise 9

Let $G = GL_n(F)$, where F is any field. Define

$$SL_n(F) = \{A \in GL_n(F) \mid \det(A) = 1\}$$

(called the special linear group). Prove that $SL_n(F) \leq GL_n(F)$.

Solution. The identity matrix I_n has determinant 1, so $SL_n(F)$ is nonempty. If $A, B \in SL_n(F)$, then the determinant identities give

$$\det(AB^{-1}) = \det(A)\det(B^{-1}) = \det(A)\det(B)^{-1} = 1 \cdot 1^{-1} = 1.$$

Thus $AB^{-1} \in SL_n(F)$. By the subgroup criterion, $SL_n(F) \leq GL_n(F)$. Equivalently, $SL_n(F)$ is the kernel of the determinant homomorphism $\det : GL_n(F) \to F^\times$. □

Section 2.1, Exercise 10

(a) Prove that if H and K are subgroups of G then so is their intersection $H \cap K$.

(b) Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup of G (do not assume the collection is countable).

Solution.

(a) The identity lies in both H and K, so $H \cap K$ is nonempty. If $x, y \in H \cap K$, then $x, y \in H$ and $x, y \in K$. Since both are subgroups, $xy^{-1} \in H$ and $xy^{-1} \in K$. Therefore $xy^{-1} \in H \cap K$, and the subgroup criterion gives $H \cap K \leq G$.

(b) Let $\{H_i\}_{i \in I}$ be any nonempty collection of subgroups of G and set

$$H = \bigcap_{i \in I} H_i.$$

The identity of G belongs to every H_i, hence to H. If $x, y \in H$, then for every $i \in I$ we have $x, y \in H_i$, and therefore $xy^{-1} \in H_i$. Consequently xy^{-1} belongs to their intersection H. The subgroup criterion proves $H \leq G$.

□

Section 2.1, Exercise 12

Let A be an abelian group and fix some $n \in \mathbb{Z}$. Prove that the following sets are subgroups of A:

(a) $\{a^n \mid a \in A\}$

(b) $\{a \in A \mid a^n = 1\}$.

Solution. Since A is abelian, the map

$$\varphi_n : A \longrightarrow A, \qquad \varphi_n(a) = a^n,$$

is a homomorphism for every integer n: for $a, b \in A$,

$$\varphi_n(ab) = (ab)^n = a^n b^n = \varphi_n(a)\varphi_n(b).$$

This exponent identity also holds for $n = 0$ and for negative n.

(a) The set $\{a^n : a \in A\}$ is precisely the image $\varphi_n(A)$. The image of a homomorphism is a subgroup, so $\{a^n : a \in A\} \leq A$.

(b) The set $\{a \in A : a^n = 1\}$ is the kernel of φ_n, hence is a subgroup of A.

□

Section 2.1, Exercise 15

Let $H_1 \leq H_2 \leq \cdots$ be an ascending chain of subgroups of G. Prove that $\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G.

Solution. Set
$$H = \bigcup_{i=1}^{\infty} H_i.$$
The identity belongs to H_1, so H is nonempty. Let $x, y \in H$. Then $x \in H_i$ and $y \in H_j$ for some positive integers i, j. Since the chain is ascending, if $k = \max\{i, j\}$, then both x and y lie in H_k. Because H_k is a subgroup,
$$xy^{-1} \in H_k \subseteq H.$$
The subgroup criterion now gives $H \leq G$. □